

Investigation of Continuous-Time Quantum Walk Via Modules of Bose-Mesner and Terwilliger Algebras

M. A. Jafarizadeh^{a,b,c} *, S. Salimi^{a,b} †

^aDepartment of Theoretical Physics and Astrophysics, Tabriz University, Tabriz 51664, Iran.

^bInstitute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran.

^cResearch Institute for Fundamental Sciences, Tabriz 51664, Iran.

February 1, 2008

*E-mail:jafarizadeh@tabrizu.ac.ir

†E-mail:shsalimi@tabrizu.ac.ir

Abstract

The continuous-time quantum walk on the underlying graphs of association schemes have been studied, via the algebraic combinatorics structures of association schemes, namely semi-simple modules of their Bose-Mesner and (reference state dependent) Terwilliger algebras. By choosing the (walk)starting site as a reference state, the Terwilliger algebra connected with this choice turns the graph into the metric space with a distance function, hence stratifies the graph into a $(d+1)$ disjoint union of strata (associate classes), where the amplitudes of observing the continuous-time quantum walk on all sites belonging to a given stratum are the same. Using the similarity of all vertices of underlying graph of an association scheme, it is shown that the transition probabilities between the vertices depend only on the distance between the vertices (kind of relations or association classes). Hence for a continuous-time quantum walk over a graph associated with a given scheme with diameter d , we have exactly $(d + 1)$ different transition probabilities (i.e., the number of strata or number of distinct eigenvalues of adjacency matrix).

In graphs of association schemes with known spectrum, namely with relevant Bose-Mesner algebras of known eigenvalues and eigenstates, the transition amplitudes and average probabilities are given in terms of dual eigenvalues of association schemes. As most of association schemes arise from finite groups, hence the continuous-time walk on generic group association schemes with real and complex representations have been studied in great details, where the transition amplitudes are given in terms of characters of groups. Further investigated examples are the walk on graphs of association schemes of symmetric S_n , Dihedral D_{2m} and cyclic groups.

Also, following Ref.[1], the spectral distributions connected to the highest irreducible representations of Terwilliger algebras of some rather important graphs, namely distance regular graphs, have been presented. Then using spectral distribution, the amplitudes of continuous-time quantum walk on strongly regular graphs such as cycle graph C_n and Johnson, and strongly regular graphs such as Petersen graphs and normal subgroup

graphs have been evaluated. Likewise, using the method of spectral distribution, we have evaluated the amplitudes of continuous-time quantum walk on symmetric product of trivial association schemes such as Hamming graphs, where their amplitudes are proportional to the product of amplitudes of constituent sub-graphs, and walk does not generate any entanglement between constituent sub-graphs.

Keywords: Continuous-time quantum walk, Association scheme, Bose-Mesner algebra, Terwilliger algebra, Spectral distribution, Distance regular graph.

PACs Index: 03.65.Ud

1 Introduction

Random walks on graphs are the bases of a number of classical algorithms. Examples include 2-SAT (satisfiability for certain types of Boolean formulas), graph connectivity, and finding satisfying assignments for Boolean formulas. It is this success of random walks that motivated the study of their quantum analogs in order to explore whether they might extend the set of quantum algorithms. This has led to a number of studies. Quantum walks on the line were examined by Nayak and Vishwanath [2], and on the cycle by Aharonov et al. [3]. The latter has also considered a number of properties of quantum walks on general graphs[1]. Two distinct types of quantum walks have been identified: for the continuous-time quantum walk a time independent Hamiltonian governs a continuous evolution of a single particle in a Hilbert space spanned by the vertices of a graph [1, 3, 4, 5] , while the discrete-time quantum walk requires a quantum coin as an additional degree of freedom in order to allow for a discrete-time unitary evolution in the space of the nodes of a graph. The connection between both types of quantum walks is not clear up to now, but in both cases different topologies of the underlying graph have been studied (see, for example, [2, 6, 7, 8, 9, 10]).

Different behavior of the quantum walk as compared to the classical random walk have been reported under various circumstances. For instance, a very promising feature of a quantum walk on a hypercube, namely an exponentially faster hitting time as compared to a classical random walk, has been presently found (numerically) by Yamasaki et al. [11] and (analytically) by Kempe [12]. Indeed, first quantum algorithms based on quantum walks which offer an (exponential) speedup over their optimal classical counterpart have been reported in Refs.[13, 14].

On the other hand, the theory of association schemes has its origin in the design of statistical experiments. The motivation came from the investigation of special kinds of partitions of the cartesian square of a set for the construction of partially balanced block designs. In this

context association schemes were introduced by R. C. Bose and K. R. Nair. Although the concept of an association scheme was introduced by Bose and Nair, the term itself was first coined by R. C. Bose and T. Shimamoto in [15]. In 1973, through the work of P. Delsarte [16] certain association schemes were shown to play a central role in the study of error correcting codes. This connection of association schemes to algebraic codes, strongly regular graphs, distance regular graphs, design theory etc., further intensified their study. Association schemes have since then become the fundamental, perhaps the most important objects in algebraic combinatorics. To this regard association schemes have for some time been studied by various people under such names as centralizer algebras, coherent configurations, Schur rings etc. Correspondingly, there are many different approaches to the study of association schemes.

A further step in the study of association schemes was their algebraization. This formulation was done by R. C. Bose and D. M. Mesner who introduced to each association scheme a matrix algebra generated by the adjacency matrices of the association scheme. This matrix algebra came to be known as the adjacency algebra of the association scheme or the Bose-Mesner algebra, after the names of the people who introduced them. The other formulation was done by P. Terwilliger, known as the Terwilliger algebra. This algebra is a finite-dimensional, semisimple and is non-commutative in general. The Terwilliger algebra has been used to study P - and Q -polynomial schemes [17], group schemes [18, 19], and Doob schemes [20].

Here in this paper, we study continuous-time quantum walk on the underlying graphs arising from association schemes, by using their algebraic combinatorics structures, namely semi-simple modules of their Bose-Mesner and (reference state dependent) Terwilliger algebras. By choosing the (walk)starting site as a reference state, the Terwilliger algebra connected with this choice turns the graph into the metric space with a distance function, and hence stratifies, the graph into a $(d+1)$ disjoint union of strata (associate classes), where the amplitudes of observing the continuous-time quantum walk on all sites belonging to a given stratum are the same. Since all vertices of underlying graph of an association scheme are similar or they have

a constant measure of similarity, therefore the transition probabilities between the vertices depend only on the distance between the vertices (kind of relations or association classes). Hence for a continuous-time quantum walk over a graph associated with a given scheme with diameter d , we have exactly $(d + 1)$ different transition probabilities (i.e., the number of strata or number of distinct eigenvalues of adjacency matrix).

In graphs of association schemes with known spectrum, namely with relevant Bose-Mesner algebras of known eigenvalues and eigenstates, the transition amplitudes and average probabilities are given in terms of dual eigenvalues of association schemes. As most of association schemes arise from finite groups, hence we have studied in great details continuous-time walk on generic group association schemes with real and complex representations, where the transition amplitudes are given in terms of characters of groups. Further more, as examples, we have investigated walk on graphs of association schemes of symmetric S_n , Dihedral D_{2m} and cyclic groups.

Also following Ref.[1], we have presented the spectral distribution connected to the highest irreducible representation of Terwilliger algebras of some rather important graphs, namely distance regular ones (since the Hilbert space of walk consists of irreducible module of Terwilliger algebra with maximal dimension). Then using the spectral distribution, we have evaluated the amplitudes of continuous-time quantum walk distance regular graphs such as cycle graph C_n and Johnson, and strongly regular graphs such as Petersen graphs and normal subgroup graphs. Likewise, using the method of spectral distribution, we have evaluated the amplitudes of continuous-time quantum walk on symmetric product of trivial association schemes such as Hamming graphs, where their amplitudes are proportional to the product of amplitudes of constituent sub-graphs, and walk does not generate any entanglement between constituent sub-graphs.

The organization of this paper is as follows. In section 2, we give a brief outline of association schemes, Bose-Mesner and Terwilliger algebras and stratification. Section 3 is devoted

to studying continuous-time quantum walk on graphs with known spectrum. In section 4, we review continuous-time quantum walk on group association scheme. In section 5, following Ref.[1, 22], we investigate continuous-time quantum walk on distance regular graphs via spectral distribution μ of the adjacency matrix A . In section 6, we calculate the amplitudes for continuous-time quantum walk on some graphs by using the prescription of sections 3, 4, 5. The paper is ended with a brief conclusion and three appendices, where the first appendix consists of studying the method of symmetrization of non-symmetric group schemes, the second appendix contains the proof of the lemma regarding the equality of the amplitudes associated with the vertices belonging to the same stratum, and the third appendix contains the list of some of the finite distance regular graphs with their corresponding spectral distributions, respectively.

2 Association scheme, Bose-Mesner algebra, Terwilliger algebra and its modules

In this section we give a brief outline of some of the main features of association scheme, such as adjacency matrices, Bose-Mesner algebra and Terwilliger algebra. At the end by choosing the (walk)starting site as a reference state we stratify the underlying graphs of association schemes via the relevant Terwilliger algebra connected with this choice.

2.1 Association schemes

First we recall the definition of association schemes. The reader is referred to Ref.[23], for further information on association schemes.

Definition 2.1. (Association schemes). Let V be a set of vertices, and let $R_i (i = 0, 1, \dots, d)$ be nonempty relations on V (i.e., subset of $V \times V$). Let the following conditions (1), (2), (3) and (4) be satisfied. Then the pair $Y = (V, \{R_i\}_{0 \leq i \leq d})$ consisting of a set V and a set of relations

$\{R_i\}_{0 \leq i \leq d}$ is called an association scheme.

(1) $\{R_i\}_{0 \leq i \leq d}$ is a partition of $V \times V$

(2) $R_0 = \{(\alpha, \alpha) : \alpha \in V\}$

(3) $R_i = R_i^t$ for $0 \leq i \leq d$, where $R_i^t = \{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$

(4) Given $(\alpha, \beta) \in R_k$, $p_{ij}^k = |\{\gamma \in V : (\alpha, \beta) \in R_i \text{ and } (\gamma, \beta) \in R_j\}|$, where the constants p_{ij}^k are called the intersection numbers, depend only on i, j and k and not on the choice of $(\alpha, \beta) \in R_k$.

Then the number n of the vertices V is called the order of the association scheme and R_i is called a relation or associate class.

Let $\Gamma = (V, R)$ denote a finite, undirected, connected graph, with vertex set V , edge set R , path-length distance function ∂ , and diameter $d := \max\{\partial(\alpha, \beta) : \alpha, \beta \in V\}$. For all $\alpha, \beta \in V$ and all integer i , we set

$$\Gamma_i = (V, R_i) = \{(\alpha, \beta) : \alpha, \beta \in V : \partial(\alpha, \beta) = i\}, \quad (2-1)$$

so that $\Gamma_i(\alpha) = \{\beta \in V : \partial(\alpha, \beta) = i\}$. Then Γ is distance regular graph if $\Gamma_0, \Gamma_1, \dots, \Gamma_d$ form an association scheme on V . In any connected graph, if $\beta \in \Gamma_i(\alpha)$, then

$$\Gamma_1(\beta) \subseteq \Gamma_{i-1}(\alpha) \cup \Gamma_i(\alpha) \cup \Gamma_{i+1}(\alpha). \quad (2-2)$$

Hence in a distance regular graph, $p_{j1}^i = 0$ (for $i \neq 0$, j is not $\{i-1, i, i+1\}$) and we set

$$a_i = p_{ii}^0, \quad b_i = p_{i-1,1}^i, \quad c_i = p_{i+1,1}^i, \quad (2-3)$$

see Fig.1.

2.2 The Bose-Mesner algebra

Let C denote the field of complex numbers. By $Mat_V(C)$ we mean the C -algebra consisting of all matrices whose entries are in C and whose rows and columns are indexed by V . For each

integer i ($0 \leq i \leq d$), let A_i denote the matrix in $Mat_V(C)$ with (α, β) -entry

$$(A_i)_{\alpha, \beta} = \begin{cases} 1 & \text{if } (\alpha, \beta) \in R_i, \\ 0 & \text{otherwise} \end{cases} \quad (\alpha, \beta \in V). \quad (2-4)$$

The matrices A_i are called the adjacency matrices of the association scheme. We then have $A_0 = I$ (by (2) above) and

$$A_i A_j = \sum_{k=0}^d p_{ij}^k A_k \quad (2-5)$$

(by (4) above), so A_0, A_1, \dots, A_d form a basis for a commutative algebra \mathbf{A} of $Mat_V(C)$, where \mathbf{A} is known as the Bose-Mesner algebra of $Y = (V, \{R_i\}_{0 \leq i \leq d})$.

Then, by using Eq.(2-3) and Bose-Mesner algebra (2-5), for adjacency matrices of distance regular graph Γ , we have

$$\begin{aligned} A_1 A_i &= c_{i-1} A_{i-1} + (a_1 - b_i - c_i) A_i + b_{i+1} A_{i+1}, \\ A_1 A_d &= c_{d-1} A_{d-1} + (a_1 - b_d) A_d, \end{aligned} \quad (2-6)$$

where A_i is a polynomial in A_1 of degree i and A_1^i is a linear combination of I, A_1, \dots, A_i .

Since the matrices A_i commute, they can be diagonalized simultaneously (see Marcus and Minc [24]), that is, there exist a matrix S such that for each $A \in \mathbf{A}$, $S^{-1}AS$ is a diagonal matrix. Therefore \mathbf{A} is semisimple and has a second basis E_0, \dots, E_d (see [23]). These are matrices satisfying

$$\begin{aligned} E_0 &= \frac{1}{n} J \\ E_i E_j &= \delta_{ij} E_i \\ \sum_{i=0}^d E_i &= I. \end{aligned} \quad (2-7)$$

The matrix $\frac{1}{n} J$ (where J is the all-one matrix in \mathbf{A}) is a minimal idempotent (idempotent is clear, and minimal follows from the $\text{rank}(J) = 1$). The E_i , for $(0 \leq i, j \leq d)$ are known as the primitive idempotent of Y . Let P and Q be the matrices relating our two bases for \mathbf{A} :

$$A_j = \sum_{i=0}^d P_{ij} E_i, \quad 0 \leq j \leq d,$$

$$E_j = \frac{1}{n} \sum_{i=0}^d Q_{ij} A_i, \quad 0 \leq j \leq d. \quad (2-8)$$

Then clearly

$$PQ = QP = nI. \quad (2-9)$$

It also follows that

$$A_j E_i = P_{ij} E_i, \quad (2-10)$$

which shows that the P_{ij} (resp. Q_{ij}) is the i -th eigenvalue (resp. the i -th dual eigenvalue) of A_j (resp. E_j) and that the columns of E_i are the corresponding eigenvectors. Thus $m_i = \text{rank}(E_i)$ is the multiplicity of the eigenvalue P_{ij} of A_j (provided that $P_{ij} \neq P_{kj}$ for $k \neq i$). We see that $m_0 = 1$, $\sum_i m_i = n$, and $m_i = \text{trace} E_i = n(E_i)_{jj}$ (indeed, E_i has only eigenvalues 0 and 1, so $\text{rank}(E_k)$ equals the sum of the eigenvalues). Also, by [25, 26], the eigenvalues and dual eigenvalues satisfy

$$\begin{aligned} P_{i0} = Q_{i0} = 1, \quad P_{0i} = k_i, \quad Q_{0i} = m_i \\ m_j P_{ji} = k_i Q_{ij}, \quad 0 \leq i, j \leq d, \end{aligned} \quad (2-11)$$

where for all integer i ($0 \leq i \leq d$), set $k_i = p_{ii}^0$, and note that $k_i \neq 0$, since R_i is non-empty.

2.3 The Terwilliger algebra and its modules

We now recall the dual Bose-Mesner algebra of Y . Given a *base vertex* $\alpha \in V$, for all integers i define $E_i^* = E_i^*(\alpha) \in \text{Mat}_V(C)$ ($0 \leq i \leq d$) to be the diagonal matrix with (β, β) -entry

$$(E_i^*)_{\beta, \beta} = \begin{cases} 1 & \text{if } (\alpha, \beta) \in R_i, \\ 0 & \text{otherwise} \end{cases} \quad (\alpha \in V). \quad (2-12)$$

The matrix E_i^* is called the i -th dual idempotent of Y with respect to α . We shall always set $E_i^* = 0$ for $i < 0$ or $i > d$. From the definition, the dual idempotents satisfy the relations

$$\sum_{i=0}^d E_i^* = I$$

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad 0 \leq i, j \leq d. \quad (2-13)$$

It follows that the matrices $E_0^*, E_1^*, \dots, E_d^*$ form a basis for the subalgebra $\mathbf{A}^* = \mathbf{A}^*(\alpha)$ of $\text{Mat}_V(R)$. \mathbf{A}^* is known as the dual Bose-Mesner algebra of Y with respect to α . For each integer i ($0 \leq i \leq d$), let $A_i^* = A_i^*(\alpha)$ denote the diagonal matrix in $\text{Mat}_V(R)$ with (β, β) -entry

$$(A_i^*)_{(\beta, \beta)} = n(E_i)_{\alpha, \beta} \quad (\beta \in V). \quad (2-14)$$

With reference to [25, 26] the matrices $A_0^*, A_1^*, \dots, A_d^*$ form a second basis for \mathbf{A}^* and satisfy

$$\begin{aligned} A_0^* &= I, \\ A_i^{*t} &= A_i^*, \\ A_0^* + A_1^* + \dots + A_d^* &= nE_0^*, \\ A_i^* A_j^* &= \sum_{h=0}^d q_{ij}^h A_h^*. \end{aligned} \quad (2-15)$$

Then by combining (2-8) with (2-12) and (2-14) we have

$$\begin{aligned} A_j^* &= \sum_{i=0}^d Q_{ij} E_i^*, \quad 0 \leq j \leq d, \\ E_j^* &= \frac{1}{n} \sum_{i=0}^d P_{ij} A_i^*, \quad 0 \leq j \leq d. \end{aligned} \quad (2-16)$$

Let $Y = (V, \{R_i\}_{0 \leq i \leq d})$ denote a scheme. Fix any $\alpha \in V$, and write $\mathbf{A}^* = \mathbf{A}^*(\alpha)$. Let $T = T(\alpha)$ denote the subalgebra of $\text{Mat}_V(C)$ generated by \mathbf{A} and \mathbf{A}^* . We call T the Terwilliger algebra of Y with respect to α .

Thus, we can define quantum decomposition for distance regular graphs by the following lemma:

Lemma (Terwilliger [17]). Let Γ denote a distance regular graph with diameter d . Fix any vertex α of Γ , and write $E_i^* = E_i^*(\alpha)$ ($0 \leq i \leq d$), $A_1 = A$ and $T = T(\alpha)$. Define $A^- = A^-(\alpha)$, $A^0 = A^0(\alpha)$, $A^+ = A^+(\alpha)$ by

$$A^- = \sum_{i=1}^d E_{i-1}^* A E_i^*, \quad A^0 = \sum_{i=1}^d E_i^* A E_i^*, \quad A^+ = \sum_{i=1}^d E_{i+1}^* A E_i^*. \quad (2-17)$$

Then

$$A = A^+ + A^- + A^0, \quad (2-18)$$

where, this is the quantum decomposition of adjacency matrix A such that,

$$(A^-)^t = A^+, \quad (A^0)^t = A^0, \quad (2-19)$$

which can be verified easily.

Let $W = C^V$ denote the vector space over C consisting of column vectors whose coordinates are indexed by V and whose entries are in C . We observe $Mat_V(C)$ which acts on W by left multiplication. We endow W with the Hermitian inner product \langle, \rangle which satisfies $\langle u, v \rangle = u^t \bar{v}$ for all $u, v \in W$, where t denotes the transpose and $-$ denotes the complex conjugation. For all $\beta \in V$, let $|\beta\rangle$ denote the element of W with a 1 in the β coordinate and 0 in all other coordinates. We observe $\{|\beta\rangle | \beta \in V\}$ is an orthonormal basis for W . Using (2-12) we have

$$E_i^* W = span\{|\beta\rangle | \beta \in V, (\alpha, \beta) \in R_i\}, \quad 0 \leq i \leq d. \quad (2-20)$$

With the use of Eq.(2-20) and since $\{|\beta\rangle | \beta \in V\}$ is an orthonormal basis for W , we get

$$W = E_0^* W \oplus E_1^* W \oplus \cdots \oplus E_d^* W, \quad (2-21)$$

where the orthogonal direct sum, with $0 \leq i \leq d$, E_i^* acts on W as the projection onto $E_i^* W$, similarly $W = \sum_{i=0}^d E_i W$. We call $E_i^* W$ the i -th subconstituent of $\Gamma = (V, R)$ (distance regular graph) with respect to α . For $0 \leq i \leq d$ we define

$$|\phi_i\rangle = \sum |\beta\rangle, \quad (2-22)$$

where the sum is over all vertices $\beta \in V$ such that $(\alpha, \beta) \in R_i$. We observe $|\phi_i\rangle \in E_0^* W$.

By a T -module we mean a subspace $U \subseteq W$ such that $TU \subseteq U$. Let U denote a T -module. Then U is said to be irreducible whenever U is nonzero and U contains no T -modules other than 0 and U . Let U denote an irreducible T -module. Then U is the orthogonal direct sum

of the nonzero spaces among $E_0^*U, E_1^*U, \dots, E_d^*U$ ([17], Lemma 3.4). By the endpoint of U we mean $\min \{i | 0 \leq i \leq d, E_i^*U \neq 0\}$. By the diameter of U we mean $|\{i | 0 \leq i \leq d, E_i^*U \neq 0\}| - 1$. We say U is thin whenever E_i^*U has dimension at most 1 for $0 \leq i \leq d$. There exists a unique irreducible T -module which has endpoint 0 ([34], Prop. 8.4). This module is called W_0 . For $0 \leq i \leq d$ the vector $|\phi_i\rangle$ of Eq.(2-22) is a basis for $E_i^*W_0$ ([17], Lemma 3.6). Therefore W_0 is thin with diameter d such that the module W_0 is orthogonal to each irreducible T -module other than W_0 ([35], Lem. 3.3).

2.4 Stratification

For $\alpha \neq \beta$ let $\partial(\alpha, \beta)$ be the length of the shortest walk connecting α and β . By definition $\partial(\alpha, \beta) = 0$ for all $\alpha \in V$. The graph becomes a metric space with the distance function ∂ . Note that $\partial(\alpha, \beta) = 1$ if and only if $\alpha \sim \beta$. We fix a point $o \in V$ as an origin of the graph. Then, the graph is stratified into a disjoint union of associate classes $\Gamma_k(o)$:

$$V = \bigcup_{k=0}^d \Gamma_k(o), \quad \Gamma_k(o) = \{\alpha \in V; \partial(o, \alpha) = k\}. \quad (2-23)$$

With each associate class $\Gamma_k(o)$ we associate a unit vector in $l^2(V)$ defined by

$$|\phi_k\rangle = \frac{1}{\sqrt{a_k}} \sum_{\alpha \in \Gamma_k(o)} |\alpha\rangle \in E_k^*W, \quad (2-24)$$

where, $|\alpha\rangle$ denotes the eigenket of α -th vertex at the associate class $\Gamma_k(o)$ and $a_k = |\Gamma_k(o)|$. Obviously, the two sites, first and last strata, have the same stratification. The closed subspace of $l^2(V)$ spanned by $\{|\phi_k\rangle\}$ is denoted by $\Lambda(G)$. Since $\{|\phi_k\rangle\}$ becomes a complete orthonormal basis of $\Lambda(G)$, we often write

$$\Lambda(G) = \sum_k \oplus \mathbf{C} |\phi_k\rangle. \quad (2-25)$$

Let A_i be the adjacency matrix of a distance regular graph $\Gamma = (V, R)$ for vacuum state $|0\rangle$ we have

$$A_k |0\rangle = \sum_{\beta \in \Gamma_k(o)} |\beta\rangle. \quad (2-26)$$

Also, we have

$$\begin{aligned} A_k|0\rangle &\in E_k^*W \\ E_k^*A_l|\phi_0\rangle &= \delta_{lk}A_l|\phi_0\rangle \end{aligned} \tag{2-27}$$

Then by using unit vectors $|\phi_k\rangle$ ($|0\rangle = |o\rangle = |\phi_0\rangle$, with $o \in V$ as the fixed origin), and Eq.(2-26), (2-27) we have

$$A_k|\phi_0\rangle = \sqrt{a_k}|\phi_k\rangle. \tag{2-28}$$

3 Continuous-time quantum walk on graphs with known spectrum

The continuous-time quantum walk is defined by replacing Kolmogorov's equation with Schrödinger's equation. Continuous-time quantum walk was introduced by Farhi and Gutmann [5] (see also [8, 30]). Our treatment, though, follow closely the analysis of Moore and Russell [30] which we review next. Let $l^2(V)$ denote the Hilbert space of C -valued square-summable functions on V . With each $\alpha \in V$ we associate a ket defined by $|\alpha\rangle$, then $\{|\alpha\rangle, \alpha \in V\}$ becomes a complete orthonormal basis of $l^2(V)$.

For $0 \leq i \leq d$ the vector $|\phi_i\rangle$ of Eq.(2-24) is a basis of $E_i^*W_0$, where W_0 is unique irreducible T - module which has endpoint 0 ([34], Prop. 8.4). Therefore, Hilbert space of continuous-time quantum walk starting from a given site corresponds to the irreducible (walk starting site related T-algebra) T - module W_0 with maximal dimension. Hence other irreducible T -modules of Terwilliger algebra T are orthogonal to Hilbert space of the walk.

Let $|\phi(t)\rangle$ be a time-dependent amplitude of the quantum process on graph Γ . The wave evolution of the quantum walk is

$$i\hbar \frac{d}{dt}|\phi(t)\rangle = H|\phi(t)\rangle, \tag{3-29}$$

where we assume $\hbar = 1$, and $|\phi_0\rangle$ is the initial amplitude wave function of the particle. The solution is given by $|\phi_0(t)\rangle = e^{-iHt}|\phi_0\rangle$. It is more natural to deal with the Laplacian of the graph, defined as $L = A - D$, where D is a diagonal matrix with entries $D_{jj} = \deg(\alpha_j)$. This is because we can view L as the generator matrix that describes an exponential distribution of waiting times at each vertex. But on s -regular graphs, $D = \frac{1}{s}I$, and since A and D commute, we get

$$e^{-itH} = e^{-it(A - \frac{1}{d}I)} = e^{-it/d}e^{-itA}. \quad (3-30)$$

This introduces an irrelevant phase factor in the wave evolution. In this paper we consider $L = A = A_1$. Then using Eq.(2-8) we have

$$|\phi_0(t)\rangle = e^{-iAt}|\phi_0\rangle = e^{-i\sum_{i=0}^d P_{i1}E_i t}|\phi_0\rangle, \quad (3-31)$$

where using the algebra of idempotents i.e., Eq.(2-7), the above amplitude of wave function can be written as

$$|\phi_0(t)\rangle = \sum_{i=0}^d e^{-iP_{i1}t}E_i|\phi_0\rangle. \quad (3-32)$$

Now using Eqs.(2-8), (2-9), (2-24) and (2-28), the matrix elements of idempotent operators between eigenstates strata and eigenstates of vertices can be calculated as

$$\langle\phi_k|E_i|\phi_0\rangle = \langle\phi_k|\frac{1}{n}\sum_{l=0}^d Q_{li}A_l|\phi_0\rangle = \frac{1}{n}\sum_{l=0}^d Q_{li}\langle\phi_k|A_l|\phi_0\rangle = \frac{\sqrt{a_k}}{n}Q_{ki}, \quad (3-33)$$

or

$$\langle\beta|E_i|\phi_0\rangle = \frac{1}{n}Q_{ki}, \quad (3-34)$$

for every $|\beta\rangle \in \Gamma_k(o)$.

Finally multiplying (3-35) by $|\beta\rangle$ and using (3-38) we get the following expression for the amplitude of observing the particle at vertex β at time t

$$\langle\beta|\phi_0(t)\rangle = \sum_{i=0}^d e^{-iP_{i1}t}\langle\beta|E_i|\phi_0\rangle = \frac{1}{n}\sum_{i=0}^d e^{-iP_{i1}t}Q_{ki}. \quad (3-35)$$

Obviously the above result indicates that the amplitudes of observing walk on vertices belonging to a given stratum are the same. Actually one can straightforwardly deduce from formula

(3-38) that the transition probabilities between the vertices depend only on the distance between the vertices (kind of relations or association classes), irrespective of which site the walk has started. This is due to the fact that in association schemes the coloring of underlying graphs or the set of relations between vertices determine thoroughly everything. Hence for a continuous-time quantum walk over a graph associated with a given scheme with diameter d , we have exactly $(d + 1)$ different transition probabilities (i.e., the number of strata or number of distinct eigenvalues of adjacency matrix).

At the end by straightforward calculation, one can evaluate the average probability for finite graphs of association schemes as

$$\bar{P}(\beta) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T P_t(\beta) dt = \frac{1}{n^2} \sum_{i=0}^d Q_{ki}^2 \quad (3-36)$$

for every $|\beta\rangle \in \Gamma_k(o)$.

4 Continuous-time quantum walk on group schemes

In this section we briefly discuss continuous-time quantum walk on group schemes with real and complex representations separately.

4.1 Group association schemes

In order to study the continuous-time quantum walk on group graphs, we need to study the group association schemes. One of the most important sources of association schemes are groups. Let G be a group acting on a finite set V . Then G has a natural action on $V \times V$ given by $g(\alpha, \beta) = (g\alpha, g\beta)$ for $g \in G$ and $\alpha, \beta \in V$. The orbits

$$\{(g\alpha, g\beta) | g \in G\} \quad (4-37)$$

of $V \times V$ are called orbitals. The group G is said to act generously transitive when for every pair $(\alpha, \beta) \in V \times V$ there is a group element $g \in G$ that exchanges α and β , that is $g\alpha = \beta$ and

$g\beta = \alpha$. When G acts generously transitive, the orbitals form the relations of that association scheme. Now, in the following, we consider the orbitals which correspond to the conjugacy classes of G . Let G be a finite group, $C_0 = \{e\}, C_1, \dots, C_d$ the conjugacy classes of G . Let $G \times G$ act on G with the action defined by $\beta(\alpha_1, \alpha_2) = \alpha_1^{-1}\beta\alpha_2$ where $\beta, \alpha_1, \alpha_2 \in G$. Then the diagonal action of $G \times G$ on $G \times G$ is given by $(\beta, \gamma)(\alpha_1, \alpha_2) = (\beta(\alpha_1, \alpha_2), \gamma(\alpha_1, \alpha_2)) = (\alpha_1^{-1}\beta\alpha_2, \alpha_1^{-1}\gamma\alpha_2)$. One can show that $(\beta_1, \beta_2), (\gamma_1, \gamma_2) \in G \times G$ belong to the same orbital of $G \times G$ if and only if $\beta_1^{-1}\beta_2, \gamma_1^{-1}\gamma_2$ belong to the same conjugacy class of G . Thus in this case the orbitals correspond to the conjugacy classes of G . For $i = 0, 1, \dots, d$ define

$$R_i = \{(\alpha, \beta) | \alpha^{-1}\beta \in C_i\}, \quad (4-38)$$

then R_i are the orbitals of $G \times G$ and hence $X(G) = (G, \{R_i\}_{0 \leq i \leq d})$ becomes a commutative association scheme and it is called the group association scheme of the finite group G [26]. We define class sum \bar{C}_i for $i = 0, 1, \dots, d$ as

$$\bar{C}_i = \sum_{\gamma \in C_i} \gamma \in CG, \quad (4-39)$$

then, for regular representation we have $\bar{C}_i|\alpha\rangle = \sum_{\gamma \in C_i} |\gamma\alpha\rangle$. Therefore in regular representation, the classes sum $\bar{C}_i (i = 0, 1, \dots, d)$ have the following matrix elements

$$(\bar{C}_i)_{\alpha, \beta} = \begin{cases} 1 & \text{if } (\alpha, \beta) \in C_i, \\ 0 & \text{otherwise} \end{cases} \quad (\alpha, \beta \in G). \quad (4-40)$$

Comparing the above matrix elements with those of adjacency matrices given in (2-4), we see that the classes sum are the corresponding adjacency matrices of group association scheme with the relation defined through conjugation. It is well known that the classes sum of finite group G form the basis of center of its CG ring which is certainly a commutative algebra, hence they are closed under multiplication defined in CG , i.e, we have

$$\bar{C}_i \bar{C}_j = \sum_{k=0}^d p_{ij}^k \bar{C}_k$$

(see details in [28]), where p_{ij}^k ($i, j, k = 0, 1, \dots, d$) are intersection numbers of the group association scheme $X(G)$ and have the following form:

$$p_{ij}^k = \frac{|C_i||C_j|}{|G|} \sum_{\chi} \frac{\chi(\alpha_i)\chi(\alpha_j)\overline{\chi(\alpha_k)}}{\chi(1)}, \quad (4-41)$$

where the sum is over all the irreducible characters χ of G [29]. Therefore, the idempotents $\{E_0, E_1, \dots, E_d\}$ of the group association scheme $X(G)$ are the projection operators of CG -module i.e.,

$$E_k = \frac{\chi_k(1)}{|G|} \sum_{\alpha \in G} \chi_k(\alpha^{-1})\alpha. \quad (4-42)$$

Thus eigenvalues of adjacency matrices of A_k , and idempotents E_k , respectively are

$$P_{ik} = \frac{d_i \kappa_k}{m_i} \chi_i(\alpha_k), \quad Q_{ik} = d_k \overline{\chi_k(\alpha_i)}, \quad (4-43)$$

where $d_j = \chi_j(1)$. The above defined group scheme is in general non-symmetric scheme and it can be symmetric provided that we choose a group whose whole irreducible representations of chosen group are real, such as symmetric group S_N . In appendix A, we have explained how to construct a symmetric group association scheme from a non-symmetric one.

A. Continuous-time walk on group schemes with real representations

In a finite group G with real conjugacy classes $C_0 = \{e\}, C_1, \dots, C_d$, i.e., $C(\alpha) = C(\alpha^{-1})$ for all $\alpha \in G$, all irreducible characters χ_i are real. Thus using (4-43) we can study continuous-time quantum walk on its underlying graph, where the amplitude of observing the particle at stratum k at time t , i.e., Eq.(3-35) reduces to

$$\langle \phi_k | \phi_0(t) \rangle = \frac{\sqrt{a_k}}{n} \sum_{i=0}^d d_i e^{\frac{-id_i \kappa_1 \chi_i(\alpha_1)t}{m_i}} \overline{\chi_i(\alpha_k)}, \quad (4-44)$$

also, the average probabilities over large times becomes

$$\bar{P}(k) = \frac{a_k}{n^2} \sum_{i=0}^d d_i^2 |\chi_i(\alpha_k)|^2, \quad k = 0, 1, \dots, d. \quad (4-45)$$

Therefore, the probability of observing the walk at starting vertex, i.e., the staying probability is

$$\bar{P}(0) = \frac{a_0}{n^2} \sum_{i=0}^d d_i^2 |\chi_i(0)|^2 = \frac{1}{n^2} \sum_{i=0}^d d_i^4. \quad (4-46)$$

As examples we will study continuous-time quantum on $G = S_n, D_{2m}$ graphs in section 6.

B. Continuous-time walk on group schemes with complex representations

In general all conjugacy classes of a given finite group are not real, hence some of its irreducible representations become complex and consequently we encounter with directed underlying graph or non-symmetric association scheme. But following instruction of appendix A we can generate a symmetric association scheme out of non-symmetric association scheme. Thus in this case, for continuous-time quantum walk, we need to use formulas (A-v) and (A-vii) of the appendix A, where the amplitude of observing the particle at stratum k at time t , i.e., Eq.(3-35) is

$$\langle \phi_k | \phi_0(t) \rangle = \begin{cases} \frac{\sqrt{a_k}}{n} \sum_{i=0}^l d_i e^{\frac{-id_i \kappa_1 \chi_i(\alpha_1)t}{m_i}} \overline{\chi_i(\alpha_k)} & \text{for real representation} \\ \frac{\sqrt{a_k}}{n} \sum_{i=l+1}^{\frac{d+l}{2}} d_i e^{\frac{-id_i \kappa_1 (\chi_i(\alpha_1) + \overline{\chi_i(\alpha_1)})t}{m_i}} (\overline{\chi_i(\alpha_k)} + \chi_i(\alpha_k)) & \text{for non-real representation.} \end{cases} \quad (4-47)$$

Also, the average probabilities are

$$\bar{P}(k) = \begin{cases} \frac{a_k}{n^2} \sum_{i=0}^l d_i^2 |\chi_i(\alpha_k)|^2 & \text{for real representation} \\ \frac{a_k}{n^2} \sum_{i=l+1}^{\frac{d+l}{2}} d_i^2 |(\chi_i(\alpha_k) + \overline{\chi_i(\alpha_k)})|^2 & \text{for complex representation.} \end{cases} \quad (4-48)$$

In this case the staying probability is

$$\bar{P}(0) = \frac{a_0}{n^2} \left(\sum_{i=0}^l d_i^2 |\chi_i(0)|^2 + 4 \sum_{i=l+1}^{\frac{d+l}{2}} d_i^2 |\chi_i(0)|^2 \right) = \frac{1}{n^2} \left(\sum_{i=0}^l d_i^4 + 4 \sum_{i=l+1}^{\frac{d+l}{2}} d_i^4 \right). \quad (4-49)$$

As an example we will study $G = C_n$ in section 6.

5 Investigation of Continuous-time quantum walk on distance regular graphs via spectral distribution of adjacency matrix

Even though in general modules of Terwilliger algebra stratifies the underlying graph of corresponding association scheme but in case of distance regular graphs, due to the existence of

raising and lowering operators given by (2-17), we can have a stratification similar to the one introduced in Ref[1]. As we will see in the following this particular kind of stratification leads to introduction of spectral distribution for adjacency matrix.

Here in this case, the strata states $|\phi_k\rangle$ are the same as defined by (2-24) of subsection 2.4, but further using Eqs.(2-6) and (2-18) one can show that, the raising and lowering operators given by (2-17) act over them as follows

$$A^+|\phi_k\rangle = \sqrt{\omega_{k+1}}|\phi_{k+1}\rangle, \quad k \geq 0 \quad (5-50)$$

$$A^-|\phi_0\rangle = 0, \quad A^-|\phi_k\rangle = \sqrt{\omega_k}|\phi_{k-1}\rangle, \quad k \geq 1 \quad (5-51)$$

$$A^0|\phi_k\rangle = (\alpha_{k+1})|\phi_k\rangle, \quad k \geq 0. \quad (5-52)$$

As mentioned in section 2, $|\phi_k\rangle$, $k = 0, 1, \dots, d$ form basis for W_0 which is the irreducible T -module with maximal dimension, therefore all basis of the irreducible T -module W_0 can be obtained by repeated action of raising operator A^+ on reference state $|\phi_0\rangle$) and we have $\omega_k = c_{k-1}b_k$, $\alpha_{k+1} = a_1 + c_k + b_k$ similar to Ref.[1]. The space W_0 equipped with set of operators (Γ, A^+, A^-, A^0) is an interacting Fock space associated with the Jacobi sequence $\{c_{k-1}b_k, a_1 - b_k - c_k, \quad k = 1, 2, \dots\}$.

Usually the spectral properties of the adjacency matrix of a graph play an important role in many branches of mathematics and physics and the spectral distribution can be generalized in various ways. In this work, following Ref.[22], the spectral distribution μ of the adjacency matrix A (where $A = A_1$) is defined as:

$$\langle A^m \rangle = \int_R x^m \mu(dx), \quad m = 0, 1, 2, \dots \quad (5-53)$$

where $\langle . \rangle$ is the mean value with respect to a the ground state $|\phi_0\rangle$, and according to Ref.[22], the $\langle A^m \rangle$ coincides with the number of m -step walks starting and terminating at o . Then the existence of a spectral distribution satisfying (5-53) is a consequence of Hamburgers theorem, see e.g., Shohat and Tamarkin [[38], Theorem 1.2].

We may apply the canonical isomorphism from the interacting Fock space (Hilbert space of continuous-time quantum walk starting from a given site, i.e., strata states or more precisely W_0 the irreducible T -module with maximal dimension starting) onto the closed linear span of the orthogonal polynomials determined by the Szegő-Jacobi sequences $(\{\omega_k\}, \{\alpha_k\})$, where for distance-regular graphs, the parameters ω_k and α_k are defined as

$$\omega_k = c_{k-1}b_k, \quad \alpha_k = a_1 - b_{k-1} - c_{k-1}, \quad (5-54)$$

where a_k, b_k and c_k have been introduced in the relations (2-3). More precisely, the spectral distribution μ under question is characterized by orthogonalizing the polynomials $\{Q_n\}$ defined recursively by

$$\begin{aligned} Q_0(x) &= 1, \quad Q_1(x) = x - \alpha_1, \\ xQ_n(x) &= Q_{n+1}(x) + \alpha_{n+1}Q_n(x) + \omega_nQ_{n-1}(x), \end{aligned} \quad (5-55)$$

for $n \geq 1$. If such a spectral distribution is unique (for distance-regular graphs is possible), the spectral distribution μ is determined by the identity:

$$G_\mu(x) = \int_R \frac{\mu(dy)}{x-y} = \frac{1}{x - \alpha_1 - \frac{\omega_1}{x - \alpha_2 - \frac{\omega_2}{x - \alpha_3 - \frac{\omega_3}{x - \alpha_4 - \dots}}}} = \frac{Q_{n-1}^{(1)}(x)}{Q_n(x)} = \sum_{l=1}^n \frac{B_l}{x - x_l}, \quad (5-56)$$

where $G_\mu(x)$ is called the Stieltjes transform and B_l is the coefficient in the Gauss quadrature formula corresponding to the roots x_l of the polynomial $Q_n(x)$ and where the polynomials $\{Q_n^{(1)}\}$ are defined recursively as

$$\begin{aligned} Q_0^{(1)}(x) &= 1, \\ Q_1^{(1)}(x) &= x - \alpha_2, \\ xQ_n^{(1)}(x) &= Q_{n+1}^{(1)}(x) + \alpha_{n+2}Q_n^{(1)}(x) + \omega_{n+1}Q_{n-1}^{(1)}(x), \end{aligned}$$

for $n \geq 1$.

Now if $G_\mu(x)$ is known, then the spectral distribution μ can be recovered from $G_\mu(x)$ by means of the Stieltjes inversion formula:

$$\mu(y) - \mu(x) = -\frac{1}{\pi} \lim_{v \rightarrow 0^+} \int_x^y \text{Im}\{G_\mu(u + iv)\} du. \quad (5-57)$$

Substituting the right hand side of (5-56) in (5-57), the spectral distribution can be determined in terms of $x_l, l = 1, 2, \dots$, the roots of the polynomial $Q_n(x)$, and Gauss quadrature constant $B_l, l = 1, 2, \dots$ as

$$\mu = \sum_l B_l \delta(x - x_l) \quad (5-58)$$

(for more details see Refs.[36, 38, 39, 40].)

Finally, using the relations(2-28) and orthogonal polynomial $P_n(x)$, where they satisfy the recursion relations (2-6) and (5-55), the other matrix elements $\langle \phi_k | A^m | \phi_0 \rangle$ can be written as

$$\langle \phi_k | A^m | \phi_0 \rangle = \frac{1}{\sqrt{a_k}} \int_R x^m P_k(x) \mu(dx), \quad m = 0, 1, 2, \dots \quad (5-59)$$

Our main goal in this paper is the evaluation of amplitude for continuous-time quantum walk by using Eq.(5-59) such that we have

$$\langle \phi_k | \phi_0(t) \rangle = \frac{1}{\sqrt{a_k}} \int_R e^{-ixt} P_k(x) \mu(dx), \quad (5-60)$$

where $\langle \phi_k | \phi_0(t) \rangle$ is the amplitude of observing the particle at level k at time t . The conservation of probability $\sum_{k=0} |\langle \phi_k | \phi_0(t) \rangle|^2 = 1$ follows immediately from Eq.(5-60) by using the completeness relation of orthogonal polynomials $P_n(x)$. Obviously evaluation of $\langle \phi_k | \phi_0(t) \rangle$ leads to the determination of the amplitudes at sites belonging to the associate scheme (stratum) $\Gamma_k(o)$. As proved in the appendix B, the walk has the same amplitude at all sites belonging to the same associated class (stratum). Also, for the finite graphs, the formula (5-60) yields

$$\langle \phi_k | \phi_0(t) \rangle = \frac{1}{\sqrt{a_k}} \sum_l B_l e^{-ix_l t} P_k(x_l), \quad (5-61)$$

where by straightforward calculation one can evaluate the average probabilities for the finite graphs which yields

$$\bar{P}(k) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\langle \phi_k | \phi_0(t) \rangle|^2 dt = \frac{1}{a_k} \sum_l B_l^2 P_k^2(x_l). \quad (5-62)$$

6 Examples

Here in subsection 6.1 we study continuous-time quantum walk on graphs with known spectrum.

6.1 Complete graph K_n

A complete graph with n vertices (denoted by K_n) is a graph with n vertices in which each vertex is connected to the others (with one edge between each pair of vertices). A complete graph is the trivial association scheme, with the intersection numbers

$$a_1 = n - 1, \quad b_1 = 1, \quad c_0 = n - 1. \quad (6-63)$$

It is straightforward to show that its adjacency matrix A has eigenvalues 1 and -1 with degeneracies

$$E_0 = \frac{1}{n}J, \quad E_1 = \frac{1}{n} \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}, \quad (6-64)$$

and using Eq.(3-35) we have

$$\langle \beta | \phi_0(t) \rangle = \begin{cases} \frac{1}{n}(e^{-it} + (n-1)e^{\frac{it}{n-1}}) & \text{for } \beta = 0 \\ \frac{-2}{n} \sin(\frac{tn}{2(n-1)}) e^{\frac{-it(n-2)}{2(n-1)}} & \text{for } \beta \neq 0. \end{cases} \quad (6-65)$$

Finally using Eq.(3-36) we get the following expressions for the average probabilities

$$P(\beta) = \begin{cases} 1 - \frac{2(n-1)}{n^2} & \text{for } \beta = 0 \\ \frac{2}{n^2} & \text{for } \beta \neq 0. \end{cases} \quad (6-66)$$

6.1.1 Symmetric group S_n

The symmetric group S_n is ambivalent in the sense that $C(\alpha) = C(\alpha^{-1})$ for all $\alpha \in S_n$, therefore its conjugacy classes form a symmetric association scheme (see Fig.2).

For group S_n , conjugacy classes are determined by the cycle structures of elements when they are expressed in the usual cycle notation. The useful notation for describing the cycle structure is the cycle type $[\nu_1, \nu_2, \dots, \nu_n]$, which is the listing of number of cycles of each length (i.e, ν_1 is the number of one cycles, ν_2 is that of two cycles and so on). Thus, the number of elements in a conjugacy class or stratum is given by

$$|C_{[\nu_1, \nu_2, \dots, \nu_n]}| = \frac{n!}{\nu_1! 2^{\nu_2} \nu_2! \dots n^{\nu_n} \nu_n!}. \quad (6-67)$$

On the other hand a partition λ of n is a sequence $(\lambda_1, \dots, \lambda_n)$ where $\lambda_1 \geq \dots \geq \lambda_n$ and $\lambda_1 + \dots + \lambda_n = n$, where in terms of cycle types

$$\lambda_1 = \nu_1 + \nu_2 + \dots + \nu_n,$$

$$\lambda_2 = \nu_2 + \nu_3 + \dots + \nu_n,$$

$$\vdots$$

$$\lambda_n = \nu_n. \quad (6-68)$$

The notation $\lambda \vdash n$ indicates that λ is a partition of n . There is one conjugacy class for each partition $\lambda \vdash n$ in S_n , which consists of those permutations having cycle structure described by λ . We denote by C_λ the conjugacy class of S_n consisting of all permutations having cycle structure described by λ . Therefore the number of conjugacy classes of S_n , namely diameter of its scheme is equal to the number of partitions of n , which grows approximately by $\frac{1}{4\pi\sqrt{3}}e^{\pi\sqrt{2n/3}}$. We consider the case where the generating set consists of the set of all transposition, i.e, $C_1 = C_{[2,1,1,1,1,\dots,1]}$. For the characters at the transposition, it is known that [31]

$$\chi_\lambda(\alpha_1) = \frac{2!(n-2)!dim(\rho_\lambda)}{n!} \sum_j \left(\binom{\lambda_j}{2} - \binom{\lambda'_j}{2} \right). \quad (6-69)$$

Here, λ' is the partition generated by transposing the Young diagram of λ , while λ'_j and λ_j are the j -th components of the partitions λ' and λ , and ρ_λ is the irreducible representation corresponding to partition λ .

Then the eigenvalues of its adjacency matrix can be written as

$$P_{\lambda 1} = \frac{d_\lambda k_1}{m_\lambda} \chi_\lambda(\alpha_1) = \sum_j \left(\begin{pmatrix} \lambda_j \\ 2 \end{pmatrix} - \begin{pmatrix} \lambda'_j \\ 2 \end{pmatrix} \right). \quad (6-70)$$

Therefore by using Eqs.(4-44) and (6-70), one can obtain the amplitudes on symmetric groups.

As an example we obtain amplitude for associate class of conjugacy class of n -cycles, as

$$\langle \phi_n | \phi_0(t) \rangle = \frac{(2i \sin(nt/2))^{n-1}}{\sqrt{nn!}}, \quad (6-71)$$

where the results thus obtained are in agreement with those of Ref[32].

In the above calculation, we have used the following results for the characters of the n -cycles

$$\chi_\lambda((n)) = \begin{cases} (-1)^{n-k} & \text{for } \lambda = (k, 1, \dots, 1), k \in \{1, \dots, n\} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\chi_{(k,1,\dots,1)}(\text{id}) = \dim(\rho_{(k,1,\dots,1)}) = \binom{n-1}{k-1}, \quad P_{\lambda 1} = \frac{1}{2}(2nk - n^2 - n).$$

6.1.2 Dihedral group D_{2m}

The dihedral group $G = D_{2m}$ is semi-direct product of cyclic groups Z_m and Z_2 with corresponding generators a and b . Hence it is generated by generators a and b with following relations:

$$D_{2m} = \langle a, b : a^m = b^2 = 1, b^{-1}ab = a^{-1} \rangle. \quad (6-72)$$

In finding its conjugacy classes, it is convenient to consider whether m is odd or even. Hence, we will study continuous-time quantum walk on dihedral group for odd and even m , separately.

1. m=odd the dihedral group D_{2m} has precisely $\frac{1}{2}(m+1)$ conjugacy classes:

$$\begin{aligned} C_0 &= \{1\}, C_1 = \{b, ab, a^2b, \dots, a^{m-1}b\}, \\ C_2 &= \{a, a^{-1}\}, \dots, C_{\frac{m+1}{2}} = \{a^{(m-1)/2}, a^{-(m-1)/2}\} \end{aligned} \quad (6-73)$$

(for more details see Ref.[28]).

By using Eqs.(4-43) and (3-35) we obtain the following amplitudes

$$\langle \phi_k | \phi_0(t) \rangle = \begin{cases} \frac{1}{m}((m-1) + \cos(mt)) & \text{for } k = 0 \\ \frac{1}{\sqrt{m}}(-i \sin(mt)) & \text{for } k = 1 \\ \frac{\sqrt{2}}{m}(\cos(mt) - 1) & \text{for } k = 2, 3, \dots, (m+3)/2. \end{cases} \quad (6-74)$$

Using Eq.(3-36) we obtain the following average transition probabilities to strata k at time t

$$\bar{P}(k) = \begin{cases} \frac{1}{m^2}((m-1)^2 + \frac{1}{2}) & \text{for } k = 0 \\ \frac{1}{2m} & \text{for } k = 1 \\ \frac{3}{m^2} & \text{for } k = 2, 3, \dots, (m+3)/2, \end{cases} \quad (6-75)$$

1. m=even the dihedral group D_{2m} ($m = 2l$) has precisely $l+3$ conjugacy classes:

$$\begin{aligned} C_0 &= \{1\}, C_1 = \{a^l\}, C_2 = \{a, a^{-1}\}, \dots, C_l = \{a^{l-1}, a^{-l+1}\} \\ C_{l+1} &= \{a^{2j}b : 0 \leq j \leq l-1\}, C_{l+2} = \{a^{2j+1}b : 0 \leq j \leq l-1\} \end{aligned} \quad (6-76)$$

(for more details see Ref.[28]). Since we will restrict our attention to C_1 generate group, therefore we consider conjugacy classes as

$$\tilde{C}_0 = C_0, \quad \tilde{C}_1 = C_{l+1} \cup C_{l+2}, \quad \tilde{C}_2 = C_1, \tilde{C}_3 = C_2, \tilde{C}_4 = C_3, \dots, \tilde{C}_{l+1} = C_l. \quad (6-77)$$

In this case, calculation of amplitudes and probabilities is similar to that of dihedral groups with odd m .

6.1.3 Cycle graph C_n

A cycle graph or cycle is a graph that consists of some number of vertices connected in a closed chain. The cycle graph with n vertices is denoted by C_n , where its graphical representation cyclic group $Z_n = \langle \alpha \rangle$, whit $\alpha^n = 1$. In this case, we consider the orbitals to correspond to the conjugacy classes of cyclic group. Also, we give $\tilde{C} = C(\alpha) \cup C(\alpha^{-1})$ for all $\alpha \in Z_n$, therefore the relations R_i form a symmetric association scheme with d classes on C_n (called the conjugacy scheme of C_n). Let $\omega_j = e^{2\pi i j/n}$ for $j = 0, 1, \dots, n-1$. Using the properties of characters of cyclic group, and Eq.(A-iv) and (A-v), we obtain

$$\begin{aligned}\tilde{P}_{j1} &= \chi_j(1) + \chi_j(n-1) = \omega_j + \omega_j^{n-1} = 2 \cos(2\pi j/n). \\ \tilde{Q}_{kj} &= \overline{\chi_k(j)} + \overline{\chi_k(n-j)} = 2 \cos(2\pi jk/n).\end{aligned}\tag{6-78}$$

Therefore by using Eq.(3-35), for strata k we have

$$\langle \phi_k | \phi_0(t) \rangle = \begin{cases} \frac{1}{n}(e^{-it} + 2 \sum_{j=0}^d e^{-it \cos(2\pi j/n)}) & \text{for } k = 0 \\ \frac{\sqrt{2}}{n}(e^{-it} + 2 \sum_{j=0}^d e^{-it \cos(2\pi j/n)} \cos(2\pi jk/n)) & \text{for } k = 1, 2, \dots, d, \end{cases}\tag{6-79}$$

where the results thus obtained are in agreement with those of Ref.[1, 33]. Thus, one can evaluate the average probability of staying at origin for large time as follows Eq.(4-49)

$$\bar{P}(0) = \frac{1}{n^2}(1 + 4 \sum_{j=1}^d \cos^2(0)) = \frac{1}{n^2}(1 + 4d),\tag{6-80}$$

and using Eq.(3-36) we get the following probabilities of transition to stratum k

$$\bar{P}(k) = \frac{2}{n^2}(1 + 4 \sum_{j=1}^d \cos^2(2\pi jk/n)).\tag{6-81}$$

Where here we have considered odd, n , and calculation for even n is similar to that of cycle graph with the odd one.

In the remaining the part of paper we will consider the examples of continuous-time quantum walk which can be investigated via spectral distributions of their corresponding adjacency matrices.

6.2 Strongly regular graphs

A graph (simple, undirected and loopless) of order n is strongly regular with parameters n, κ, λ, η whenever it is not complete or edgeless and

- (i) each vertex is adjacent to κ vertices,
- (ii) for each pair of adjacent vertices there are λ vertices adjacent to both,
- (iii) for each pair of non-adjacent vertices there are η vertices adjacent to both.

For strongly regular graph, the intersection numbers are given by

$$a_1 = \kappa; \quad b_1 = 1, \quad b_2 = \eta; \quad c_0 = \kappa, \quad c_1 = \kappa - \lambda - 1. \quad (6-82)$$

By using formula (5-58) one can straightforwardly get the following spectral distribution

$$\mu = B_1 \delta(x - x_1) + B_2 \delta(x - x_2) + B_3 \delta(x - x_3), \quad (6-83)$$

where, we obtain x_i and B_i for $i = 1, 2, 3$ respectively as

$$\begin{aligned} x_1 &= \kappa, \\ x_2 &= \frac{1}{2}(\lambda - \eta + \sqrt{(\lambda - \eta)^2 - 4(\eta - \kappa)}), \\ x_3 &= \frac{1}{2}(\lambda - \eta - \sqrt{(\lambda - \eta)^2 - 4(\eta - \kappa)}), \\ B_1 &= \frac{\eta}{\kappa^2 - \kappa(\lambda - \eta) + (\eta - \kappa)}, \\ B_2 &= \frac{-\kappa\sqrt{(\lambda - \eta)^2 - 4(\eta - \kappa)} + \kappa(\lambda - \eta) + 2\kappa}{(\lambda - \eta - 2\kappa)\sqrt{(\lambda - \eta)^2 - 4(\eta - \kappa)} + (\lambda - \eta)^2 - 4(\eta - \kappa)}, \\ B_3 &= \frac{\kappa\sqrt{(\lambda - \eta)^2 - 4(\eta - \kappa)} + \kappa(\lambda - \eta) + 2\kappa}{(-\lambda + \eta + 2\kappa)\sqrt{(\lambda - \eta)^2 - 4(\eta - \kappa)} + (\lambda - \eta)^2 - 4(\eta - \kappa)}. \end{aligned} \quad (6-84)$$

$$(6-85)$$

Again using Eq.(5-60) one can obtain the amplitudes for quantum walk at strata k and time t .

For example we study continuous-time quantum walk on the following graphs.

6.2.1 Petersen graph

Petersen graph is strongly regular graph with parameters $(n, \kappa, \lambda, \eta) = (10, 3, 0, 1)$ (see Fig.3).

The intersection numbers and spectral distribution are

$$a_1 = 3, \quad a_2 = 6; \quad b_1 = b_2 = 1; \quad c_0 = 3, \quad c_1 = 2.$$

$$\mu = \frac{1}{10}\delta(x-3) + \frac{1}{2}\delta(x-1) + \frac{2}{5}\delta(x+2). \quad (6-86)$$

Therefore, the amplitudes for walk at time t are

$$\begin{aligned} \langle \phi_0 | \phi_0(t) \rangle &= \int_R e^{-ixt} \mu(dx) = \frac{1}{2}e^{-it} + \frac{2}{5}e^{2it} + \frac{1}{10}e^{-i3t} \\ \langle \phi_1 | \phi_0(t) \rangle &= \frac{1}{\sqrt{3}} \int_R x e^{-ixt} \mu(dx) = \frac{1}{\sqrt{3}} \left(\frac{1}{2}e^{-it} - \frac{4}{5}e^{2it} + \frac{3}{10}e^{-i3t} \right) \\ \langle \phi_2 | \phi_0(t) \rangle &= \frac{1}{\sqrt{6}} \int_R (x^2 - 3) e^{-ixt} \mu(dx) = \frac{1}{\sqrt{6}} \left(-e^{-it} + \frac{2}{5}e^{2it} + \frac{2}{5}e^{-i3t} \right). \end{aligned} \quad (6-87)$$

6.2.2 Normal subgroup graphs

Let G be a finite group, and $P = \{P_0, P_1, \dots, P_d\}$ be a blueprint of it. We always assume that the sets P_i are so numbered that the identity element e of G belongs to P_0 . If $P_0 = \{e\}$, then P is called homogeneous. Let $\{R_0, R_1, \dots, R_d\}$ be the set of relations $R_i = \{(\alpha, \beta) \in G \otimes G | \alpha^{-1}\beta \in P_i\}$ on G . Now, we define a blueprint for group G which form a strongly regular graph. If H is a subgroup of G , we define the blueprints by

$$P_0 = \{e\}, \quad P_1 = G - \{H\}, \quad P_2 = H - \{e\}. \quad (6-88)$$

This blueprint form a strongly regular graph with parameters $(n, \kappa, \lambda, \eta) = (|G|, |G| - |H|, |G| - 2|H|, |G| - |H|)$.

As an example, we consider $G = D_{2m}$:

1. m=odd in this case the subgroup is defined as

$$H = \{e, a, a^{-1}, \dots, a^{(m-1)/2}, a^{-(m-1)/2}\}. \quad (6-89)$$

Therefore the blueprints are given by

$$\begin{aligned} P_0 &= \{e\}, \quad P_1 = \{b, ab, a^2b, \dots, a^{m-1}b\}, \\ P_2 &= \{a, a^{-1}, \dots, a^{(m-1)/2}, a^{-(m-1)/2}\}, \end{aligned} \quad (6-90)$$

which form a strongly regular graph with parameters $(2m, m, 0, m)$. By using Eqs.(6-82),(6-83), (6-84) and (6-85), we get the following expressions for the intersection numbers and spectral distribution

$$\begin{aligned} a_1 &= m, \quad a_2 = m - 1; \quad b_1 = 1, \quad b_2 = m; \quad c_0 = m, \quad c_1 = m - 1. \\ \mu &= \frac{1}{2m}\delta(x - m) + \frac{m - 1}{m}\delta(x) + \frac{1}{2m}\delta(x + m). \end{aligned} \quad (6-91)$$

Therefore, the amplitudes for walk at time t are

$$\begin{aligned} \langle \phi_0 | \phi_0(t) \rangle &= \frac{1}{m}((m - 1) + \cos(mt)) \\ \langle \phi_1 | \phi_0(t) \rangle &= \frac{1}{\sqrt{m}}(-i \sin(mt)) \\ \langle \phi_2 | \phi_0(t) \rangle &= \frac{\sqrt{m - 1}}{m}(\cos(mt) - 1). \end{aligned} \quad (6-92)$$

Also using Eq.(5-62) we evaluate the average probabilities as

$$\bar{P}(k) = \begin{cases} \frac{1}{m^2}((m - 1)^2 + \frac{1}{2}) & \text{for } k = e \\ \frac{1}{2m} & \text{for } k = 1 \\ \frac{3(m-1)}{2m^2} & \text{for } k = 2. \end{cases} \quad (6-93)$$

2. m=even we consider $m = 2l$ and define subgroup as

$$H = \{e, a, a^{-1}, \dots, a^{(l-1)}, a^{-(l-1)}, a^l\}. \quad (6-94)$$

Therefore the blueprints are given by

$$\begin{aligned} P_0 &= \{e\}, \quad P_1 = \{a^{2j}b, a^{2j+1}b, \quad 0 \leq j \leq l - 1\}, \\ P_2 &= \{a, a^{-1}, \dots, a^{(l-1)}, a^{-(l-1)}, a^l\}, \end{aligned} \quad (6-95)$$

which are strongly regular graphs with parameters $(2m, m, 0, m)$. In this case, the calculation is similar to that of dihedral groups with m , odd.

6.3 Cycle graph C_n

spectral distribution

A cycle graph or cycle has already been defined in the subsection 6.1. The cycle graph C_n is the distance regular graph, and quantum walk on them turns out to be different for odd and even n , hence, below we treat them separately.

Odd n . For odd $n = 2d + 1$, the diameter of graph C_{2d+1} is d also the intersection numbers as

$$a_1 = a_2 = \cdots = a_d = 2; \quad b_1 = b_2 = \cdots = b_d = 1; \quad c_0 = 2, \quad c_1 = c_2 = \cdots = c_{d-1} = 1. \quad (6-96)$$

In this case, the adjacency matrices are given by

$$A_0 = I_n, \quad A_1 = S^1 + S^{-1}, \quad A_2 = S^2 + S^{-2}, \dots, A_d = S^d + S^{-d}, \quad (6-97)$$

where S is the shift operator $S^{2d+1} = 1$. Now by using the Bose-Mesner algebra of distance-regular graph for adjacency matrices (2-6), we obtain

$$A_l = 2T_l(A_1/2), \quad l \geq 1, \quad (6-98)$$

where T_l is Tchebichef polynomials of first kind. Also, Eq.(6-98) and Eq.(2-6) for $i = d$ we have

$$T_d(A_1/2) = T_{d+1}(A_1/2). \quad (6-99)$$

The canonical isomorphism from the interacting Fock space onto the closed linear span of the orthogonal polynomials maps states $|\phi_l\rangle$ into orthogonal polynomials $P_l(x) = 2T_l(x/2)$.

It is straightforward to show that the spectral distribution Eq.(5-58) as

$$\mu = \frac{1}{2d+1} \delta(x-2) + \frac{2}{2d+1} \sum_{l=1}^d \delta(x - 2 \cos(\frac{2l\pi}{2d+1})) \quad (6-100)$$

where $x_l = 2 \cos(\frac{2l\pi}{2d+1})$ are the roots of Eq.(6-99). Therefore, the amplitudes for observing the walk at time t and k th associated class is

$$\langle \phi_k | \phi_0(t) \rangle = \frac{2}{\sqrt{a_k}} \int_R e^{-ixt/2} T_k(x/2) \mu(dx)$$

$$= \frac{\sqrt{2}}{2d+1} (e^{-it} + 2 \sum_{l=1}^d e^{-it \cos 2l\pi/(2d+1)} \cos(2kl\pi/(2d+1))), \quad (6-101)$$

where the results thus obtained are in agreement with those of Refs.[1, 33].

Even n. For even $n = 2d$, the diameter of graph C_{2d} is d and the intersection numbers are

$$\begin{aligned} a_1 = a_2 = \cdots = a_{d-1} = 2, \quad a_d = 1, \quad b_1 = b_2 = \cdots = b_{d-1} = 1, \quad b_d = 2, \\ c_0 = 2, \quad c_1 = c_2 = \cdots = c_{d-1} = 1. \end{aligned} \quad (6-102)$$

In this case, its calculation is similar to that of cycle graph with n , odd.

In the limit of large n , the cycle graph C_n is the same as the infinite line graph, with the intersection numbers

$$a_1 = a_2 = \cdots = 2; \quad b_1 = b_2 = \cdots = 1; \quad c_0 = 2, \quad c_1 = c_2 = \cdots = 1. \quad (6-103)$$

Then the orthogonal polynomials are $P_l(x) = 2T_l(x/2)$ and the spectral distribution μ is

$$\mu = \frac{1}{\pi} \frac{1}{\sqrt{4-x^2}}, \quad -2 \leq x \leq 2, . \quad (6-104)$$

Then one can obtain spectral distribution and amplitude for every vertex at the time t , where the results thus obtained are in agreement with those of Ref.[1].

6.4 Johnson graph

Let v, d be a pair of positive integers such that $d \leq v$. Put $S = \{1, 2, \dots, v\}$ and $V = \{x \subset S : |x| = d\}$. We say that $x, y \in V$ are adjacent if $d - |x \cap y| = 1$. Thus a graph structure is introduced in V , which is called a Johnson graph and denoted by $J(v, d)$ (see Fig.4). By symmetry we may assume that $2d \leq v$. Consider the growing family of Johnson graphs $J(v, d)$, where $d \rightarrow \infty$ and $\frac{2d}{v} \rightarrow p \in (0, 1]$. Then the associated orthogonal polynomials are (for more details see Ref.[36])

A. for $p = 1$, we have Laguerre polynomials $L_n(x)$ with the following recurrence formula:

$$L_0(x) = 1,$$

$$L_1(x) = x - 1,$$

$$xL_n(x) = L_{n+1}(x) + (2n+1)L_n(x) + n^2L_{n-1}(x), \quad n \geq 1. \quad (6-105)$$

By using the fact that the Laguerre polynomials are orthogonal polynomials with respect to the spectral distribution $e^{-x}dx$ and following paper [1] we obtain the following amplitudes

$$\langle \phi_k | \phi_0(t) \rangle = \frac{(it)^k}{(1+it)^{k+1}}. \quad (6-106)$$

B. for $0 \leq p \leq 1$, by modifying the Meixner polynomials $M_n(x)$, we have the recurrence formula:

$$M_0(x) = 1,$$

$$M_1(x) = x,$$

$$xM_n(x) = M_{n+1}(x) + \frac{2n}{\sqrt{p(2-p)}}M_n(x) + n^2M_{n-1}(x), \quad n \geq 1. \quad (6-107)$$

Hence by using the fact that the Meixner polynomials are orthogonal polynomials with respect to the spectral distribution $\sum_{k=0}^{\infty} \frac{2(1-p)}{2-p} \left(\frac{p}{2-p}\right)^k \delta\left(x - \frac{-p+2(1-p)k}{\sqrt{p(2-p)}}\right)$ and Eq.(5-60) one can obtain the amplitudes. As an example, we obtain the amplitude at the origin at time t as

$$\langle \phi_0 | \phi_0(t) \rangle = \sum_{k=0}^{\infty} \frac{2(1-p)}{2-p} \left(\frac{p}{2-p}\right)^k e^{-i \frac{-p+2(1-p)k}{\sqrt{p(2-p)}} t}. \quad (6-108)$$

6.5 Product of association schemes

In this section, we recall some basic facts about the symmetric product of trivial schemes. (see [37] for more details.) This product is important not only as a means of constructing new association schemes from the old ones, but also for describing the structure of certain schemes in term of particular sub-schemes or schemes whose structure may already be known. Then using Eq.(3-35) and (5-60), we can evaluate amplitudes of continuous-time quantum walk on new association schemes. The symmetric product of d -tuples of trivial scheme K_n with

adjacency matrices of $I_n, J_n - I_n$ is association scheme with the following adjacency matrices (generators of its Bose-Mesner algebra)

$$\begin{aligned}
 A_0 &= I_n \otimes I_n \otimes \dots \otimes I_n, \\
 A_1 &= \sum_{\text{permutation}} (J_n - I_n) \otimes I_n \otimes \dots \otimes I_n, \\
 &\vdots \\
 A_i &= \sum_{\text{permutation}} \underbrace{(J_n - I_n) \otimes (J_n - I_n) \dots \otimes (J_n - I_n)}_i \otimes I_n \otimes \dots \otimes I_n, \quad (6-109)
 \end{aligned}$$

where J_n is $n \times n$ matrix with all matrix elements equal to one. This scheme is the well known Hamming scheme with intersection number

$$\begin{aligned}
 a_i &= \frac{(n-1)^i d(d-1) \dots (d-i+1)}{i!}, \quad 1 \leq i \leq d, \\
 b_i &= i, \quad 1 \leq i \leq d, \\
 c_i &= (n-1)(d-i), \quad 0 \leq i \leq d-1, \quad (6-110)
 \end{aligned}$$

where its underlying graph is the cartesian product of d -tuples of cyclic group Z_n . Following Ref. [1], the amplitudes of walk and the spectral distribution in the symmetric product of graphs can be obtained in terms of sub-graphs. Finally we obtain the following expression for the amplitude at origin and spectral distribution

$$\begin{aligned}
 \mu &= \sum_{l=0}^d \frac{(n-1)^{d-l} d!}{n^d l! (d-l)!} \delta(x - nl + d), \\
 \langle \phi_0 | \phi_0(t) \rangle &= \sum_{l=0}^d \frac{(n-1)^{d-l} d!}{n^d l! (d-l)!} e^{-it(nl-d)}, \quad (6-111)
 \end{aligned}$$

respectively.

Also one can show that its idempotents $\{E_0, E_1, \dots, E_d\}$ are symmetric product of d -tuples of corresponding idempotents of trivial schemes K_n . That is, we have

$$E_0 = \frac{J_n}{n} \otimes \frac{J_n}{n} \otimes \dots \otimes \frac{J_n}{n},$$

$$\begin{aligned}
 E_1 &= \sum_{\text{permutation}} \left(I_n - \frac{J_n}{n} \right) \otimes \frac{J_n}{n} \otimes \dots \otimes \frac{J_n}{n}, \\
 &\vdots \\
 E_i &= \sum_{\text{permutation}} \underbrace{\left(I_n - \frac{J_n}{n} \right) \otimes \left(I_n - \frac{J_n}{n} \right) \dots \otimes \left(I_n - \frac{J_n}{n} \right)}_i \otimes \frac{J_n}{n} \otimes \dots \otimes \frac{J_n}{n}.
 \end{aligned} \tag{6-112}$$

Therefore, for the eigenvalues P_{ij} , and dual ones Q_{ij} we get

$$\begin{aligned}
 P_{ij} &= C_i^d (C_j^d)^{-1} (n-1)^{i-j} K_j(i), \\
 Q_{ij} &= \frac{m_j}{k_i} C_j^d (C_i^d)^{-1} (n-1)^{j-i} K_i(j),
 \end{aligned} \tag{6-113}$$

where $K_k(x)$ is the Krawtchouk polynomials defined as

$$K_k(x) = \sum_{i=1}^k C_i^x C_{k-i}^{n-x} (-1)^i (d-1)^{k-i}, \tag{6-114}$$

and $C_k^l = \frac{l!}{k!(l-k)!}$. Then by using Eq.(2-10) we obtain the amplitude as

$$\langle \phi_k | \phi_0(t) \rangle = \frac{\sqrt{a_k}}{n^d} \sum_{j=0} e^{-it \frac{j(d-1)!(n-1)^{j-1}}{j!(d-j)!}} Q_{kj}. \tag{6-115}$$

7 Conclusion

Continuous-time quantum walk on underlying graphs of the association schemes have been studied by using the irreducible modules of Bose-Mesner and Terwilliger algebras connected with them, where the irreducible modules of Terwilliger algebra and dual eigenvalues of association schemes play an important role. Also Continuous-time quantum walk on distance and strongly regular graphs are investigated by using the spectral distribution associated to irreducible representation of associated Terwilliger algebras. Although the powerful method of spectral distribution associated to irreducible representation of Terwilliger algebra seems to work for continuous-time quantum walk on distance regular graphs, we expect that one can further develop it for some non-distance regular graphs of association schemes and discrete-time quantum walk on graphs of association schemes. These are under investigation. Also, it

seems that using this formalism one can study the continuous-time and discrete-time quantum walk on some graphs which lack association scheme structure, but they have the same stratification structure as those of scheme graphs, provided that quantum walk starts from a distinguished site (see Ref.[1]).

Appendix A

In this appendix, we study the method of symmetrization of group schemes of non-symmetric one. If we have $\alpha \in C_i$ but α^{-1} is not in C_i , then the association scheme is non-symmetric. In order to construct a symmetric group scheme from a give non-symmetric one, we need to define the following classes sum

$$\tilde{C}_i = \begin{cases} \bar{C}_i & \text{for } i = 0, 1, \dots, l, \\ \bar{C}_i + \bar{C}_i^{-1} & \text{for } i = l+1, \dots, \frac{d+1+l}{2}, \end{cases} \quad (\text{A-i})$$

where, \tilde{C}_i and \bar{C}_i for $i = 0, 1, \dots, l$ are real and $i = l+1, \dots, \frac{d+1+l}{2}$, are complex.

One can show that the above defined classes sum yield the following relations among themselves

$$\tilde{C}_i \tilde{C}_j = \frac{|\tilde{C}_i| |\tilde{C}_j|}{|G|} \left(\sum_{\nu, k=0}^l \frac{\chi_\nu(\alpha_i) \chi_\nu(\alpha_j) \overline{\chi_\nu(\alpha_k)}}{\chi_\nu(1)} \tilde{C}_k + \frac{1}{2} \sum_{\nu, r=l+1}^{\frac{d+1+l}{2}} \frac{\chi_\nu(\alpha_i) \chi_\nu(\alpha_j) (\overline{\chi_\nu(\alpha_r)} + \chi_\nu(\alpha_r))}{\chi_\nu(1)} \tilde{C}_r \right)$$

for $i, j = 0, 1, \dots, l$,

$$\begin{aligned} \tilde{C}_i \tilde{C}_j = \frac{|\tilde{C}_i| |\tilde{C}_j|}{2|G|} & \left(\sum_{\nu, k=0}^l \frac{\chi_\nu(\alpha_i) (\chi_\nu(\alpha_j) + \overline{\chi_\nu(\alpha_j)}) \overline{\chi_\nu(\alpha_k)}}{\chi_\nu(1)} \tilde{C}_k + \right. \\ & \left. \sum_{\nu, r=l+1}^{\frac{d+1+l}{2}} \frac{\chi_\nu(\alpha_i) (\chi_\nu(\alpha_j) + \overline{\chi_\nu(\alpha_j)}) (\overline{\chi_\nu(\alpha_r)} + \chi_\nu(\alpha_r))}{\chi_\nu(1)} \tilde{C}_r \right) \end{aligned}$$

for $i = 0, 1, \dots, l$ and $j = l+1, \dots, \frac{d+1-l}{2}$,

$$\tilde{C}_i \tilde{C}_j = \frac{|\tilde{C}_i| |\tilde{C}_j|}{2|G|} \left(\sum_{\nu, k=0}^l \frac{(\chi_\nu(\alpha_i) + \overline{\chi_\nu(\alpha_i)}) (\chi_\nu(\alpha_i) + \overline{\chi_\nu(\alpha_i)}) \overline{\chi_\nu(\alpha_k)}}{\chi_\nu(1)} \tilde{C}_k + \right.$$

$$\sum_{\nu, r=l+1}^{\frac{d+1+l}{2}} \frac{(\chi_\nu(\alpha_i) + \overline{\chi_\nu(\alpha_i)})(\chi_\nu(\alpha_i) + \overline{\chi_\nu(\alpha_i)})(\overline{\chi_\nu(\alpha_r)} + \chi_\nu(\alpha_r))}{\chi_\nu(1)} \tilde{C}_r \quad (\text{A-ii})$$

for $i, j = l+1, \dots, \frac{d+1+l}{2}$. Therefore, the corresponding idempotents $\{\tilde{E}_0, \tilde{E}_1, \dots, \tilde{E}_{\frac{d+1+l}{2}}\}$ of group association scheme are its irreducible CG -modules projection operators, i.e

$$\tilde{E}_k = \begin{cases} \frac{\chi_k(1)}{|G|} \sum_{j=0}^l \overline{\chi_k(\alpha_j)} \tilde{C}_j \\ \frac{\chi_k(1)}{|G|} \sum_{j=l+1}^{\frac{d+1+l}{2}} (\overline{\chi_k(\alpha_j)} + \chi_k(\alpha_j)) \tilde{C}_j \end{cases} \quad (\text{A-iii})$$

for $k = 0, 1, \dots, l$ and

$$\tilde{E}_k = \begin{cases} \frac{\chi_k(1)}{|G|} \sum_{j=0}^l 2\overline{\chi_k(\alpha_j)} \tilde{C}_j \\ \frac{\chi_k(1)}{|G|} \sum_{j=l+1}^{\frac{d+1+l}{2}} (\overline{\chi_k(\alpha_j)} + \chi_k(\alpha_j)) \tilde{C}_j \end{cases} \quad (\text{A-iv})$$

for $k = l+1, \dots, \frac{d+1+l}{2}$.

Obviously the above defined association scheme is symmetric. It is rather straightforward to see that its eigenvalues \tilde{P}_{ik} and dual ones \tilde{Q}_{ik} are

$$\tilde{P}_{ik} \longrightarrow \begin{cases} \frac{d_i \kappa_k}{m_i} \chi_i(\alpha_k) & \text{for } i = 0, 1, \dots, l, \\ \frac{d_i \kappa_k}{m_i} (\chi_i(\alpha_k) + \overline{\chi_i(\alpha_k)}) & \text{for } i = l+1, \dots, \frac{d+1+l}{2} \end{cases} \quad (\text{A-v})$$

for $k = 0, 1, \dots, l$ and

$$\tilde{P}_{ik} \longrightarrow \begin{cases} \frac{d_i \kappa_k}{m_i} (\chi_i(\alpha_k) + \overline{\chi_i(\alpha_k)}) & \text{for } i = 0, 1, \dots, l, \\ 2 \frac{d_i \kappa_k}{m_i} (\chi_i(\alpha_k) + \overline{\chi_i(\alpha_k)}) & \text{for } i = l+1, \dots, \frac{d+1+l}{2} \end{cases} \quad (\text{A-vi})$$

for $k = 0, 1, \dots, \frac{d+1+l}{2}$,

$$\tilde{Q}_{ik} \longrightarrow \begin{cases} d_k \overline{\chi_k(\alpha_i)} & \text{for } i = 0, 1, \dots, l, \\ d_k (\overline{\chi_k(\alpha_i)} + \chi_k(\alpha_i)) & \text{for } i = l+1, \dots, \frac{d+1+l}{2}, \end{cases} \quad (\text{A-vii})$$

for $k = 0, 1, \dots, l$ and

$$\tilde{Q}_{ik} \longrightarrow \begin{cases} 2d_k \overline{\chi_k(\alpha_i)} & \text{for } i = 0, 1, \dots, l, \\ d_k (\overline{\chi_k(\alpha_i)} + \chi_k(\alpha_i)) & \text{for } i = l+1, \dots, \frac{d+1+l}{2}, \end{cases} \quad (\text{A-viii})$$

for $k = 0, 1, \dots, \frac{d+1+l}{2}$.

In fact, eigenvalues(dual eigenvalues) are sum of real and non-real contributions.

In section 5, using the above prescription we have studied the continuous-time quantum walk on cyclic groups.

Appendix B

In this appendix we prove the following lemma in connection with the equality of continuous-time quantum walk amplitudes on the vertices belonging to the same associated class.

Lemma 1. Let $q_{ik}(t)$ denote the amplitude of observing the continuous-time quantum walk at vertex $i \in \Gamma_k(\alpha)$ at time t . Then for a class of distance-regular graphs, the amplitude q_{ik} is the same for all vertices of associated class k , for all t .

Proof.

As the distance-regular graphs form an association scheme, therefore they have corresponding Terwilliger algebra. Also, for Terwilliger algebra T , there exist a unique irreducible T -module which has reference point 0, where we called W_0 , such that the unit vector Eq.(2-24) is a basis for $E_k^*W_0$, and the irreducible T -module W_0 is orthogonal to its other irreducible T -modules. Actually the basis of irreducible T -module W_0 can be obtained simply by acting the operator A^+ on reference state $|\phi_0\rangle$ repeatedly and also the other irreducible T -modules can be obtained by repeated action of the operator A^+ over some set of orthogonal vectors which are orthogonal to irreducible T -module W_0 and at the same time they vanish under the action of the operator A^- . The remaining part of proof is almost similar to the proof presented in Appendix A of Ref. [1].

Appendix C

List of some important distance regular graphs

1. Collinearity graph gen. octagon $GO(s, t)$.

$$\mu = \frac{1}{(s+1)(st+1)(s^2t^2+1)}\delta(x-s(t+1)) + \frac{st(t+1)}{4(st+1-\sqrt{2st})(s+t+\sqrt{2st})}\delta(x-s+1-\sqrt{2st}) + \frac{st(t+1)}{2(st+1)(s+t)}\delta(x-$$

$$s+1) + \frac{st(t+1)}{4(st+1-\sqrt{2st})(s+t+\sqrt{2st})}\delta(x-s+1+\sqrt{2st}) + \frac{s^4}{(s+1)(s+t)(s^2+t^2)}\delta(x+t+1).$$

Intersection numbers:

$$c_0 = s(t+1), \quad c_1 = st, \quad c_2 = st, \quad c_3 = st,$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 1, \quad b_4 = t+1.$$

2. Collinearity graph gen. dodecagon GD(s, 1).

$$\mu = \frac{1}{((s+1)^2-3s)((s+1)^2-s)(s+1)^2}\delta(x-2s) + \frac{s-1+\sqrt{3s}}{12((s+1)^2-3s)}\delta(x-s+1-\sqrt{3s}) + \frac{s-1-\sqrt{3s}}{12((s+1)^2-3s)}\delta(x-s+1+\sqrt{3s}) + \frac{s-1+\sqrt{s}}{4((s+1)^2-s)}\delta(x-s+1+\sqrt{s}) + \frac{s-1-\sqrt{s}}{4((s+1)^2-s)}\delta(x-s+1-\sqrt{s}) + \frac{s^5}{((s+1)^2-3s)((s+1)^2-s)(s+1)^2}\delta(x+2).$$

Intersection numbers:

$$c_0 = 2s, \quad c_1 = s, \quad c_2 = s, \quad c_3 = s, \quad c_4 = s, \quad c_5 = s,$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 1, \quad b_4 = 1, \quad b_5 = 1, \quad b_6 = 2.$$

3. M_{22} graph.

$$\mu = \frac{7}{110}\delta(x+4) + \frac{3}{10}\delta(x+3) + \frac{7}{15}\delta(x-1) + \frac{1}{6}\delta(x-4) + \frac{1}{330}\delta(x-7).$$

Intersection numbers:

$$c_0 = 7, \quad c_1 = 6, \quad c_2 = 4, \quad c_3 = 4,$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 1, \quad b_4 = 6.$$

4. Incidence graph $\text{pg}(k-1, k-1, k-1)$, $k = 4, 5, 7, 8$.

$$\mu = \frac{1}{2k^2}(\delta(x-k) + \delta(x+k)) + \frac{k-1}{k^2}\delta(x) + \frac{k-1}{2k}(\delta(x-\sqrt{k}) + \delta(x+\sqrt{k})).$$

Intersection numbers:

$$c_0 = k, \quad c_1 = k-1, \quad c_2 = k-1, \quad c_3 = 1,$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = k-1, \quad b_4 = k.$$

5. Coset graph doubly truncated binary Golay.

$$\mu = \frac{21}{512}\delta(x+11) + \frac{35}{64}\delta(x+3) + \frac{105}{256}\delta(x-5) + \frac{1}{512}\delta(x-21).$$

Intersection numbers:

$$c_0 = 21, \quad c_1 = 20, \quad c_2 = 16,$$

$$b_1 = 1, \quad b_2 = 2, \quad b_3 = 12.$$

6. Coset graph extended ternary Golay code.

$$\mu = \frac{8}{243}\delta(x+12) + \frac{440}{729}\delta(x+3) + \frac{88}{243}\delta(x-6) + \frac{1}{729}\delta(x-24).$$

Intersection numbers:

$$c_0 = 24, \quad c_1 = 22, \quad c_2 = 20,$$

$$b_1 = 1, \quad b_2 = 2, \quad b_3 = 12.$$

7. Wells graph.

$$\mu = \frac{1}{16}\delta(x+3) + \frac{1}{4}(\delta(x+\sqrt{5}) + \delta(x-\sqrt{5}) + \delta(x-1)) + \frac{3}{16}\delta(x-5).$$

Intersection numbers:

$$c_0 = 5, \quad c_1 = 4, \quad c_2 = 1, \quad c_3 = 1,$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 4, \quad b_4 = 5.$$

8. 3-Cover GQ(2, 2).

$$\mu = \frac{1}{9}\delta(x+3) + \frac{2}{5}\delta(x+2) + \frac{4}{15}\delta(x-3) + \frac{1}{5}\delta(x-1) + \frac{1}{45}\delta(x-6).$$

Intersection numbers:

$$c_0 = 6, \quad c_1 = 4, \quad c_2 = 2, \quad c_3 = 1,$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 4, \quad b_4 = 6.$$

9. Double Hoffman-Singleton.

$$\mu = \frac{7}{25}(\delta(x+2) + \delta(x-2)) + \frac{21}{100}(\delta(x+3) + \delta(x-3)) + \frac{1}{100}(\delta(x+7) + \delta(x-7)).$$

Intersection numbers:

$$c_0 = 7, \quad c_1 = 6, \quad c_2 = 6, \quad c_3 = 1, \quad c_4 = 1,$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 6, \quad b_4 = 6, \quad b_5 = 7.$$

10. Foster graph.

$$\mu = \frac{1}{9}\delta(x) + \frac{1}{5}(\delta(x+1) + \delta(x-1)) + \frac{1}{10}(\delta(x+2) + \delta(x-2)) + \frac{1}{90}(\delta(x+3) + \delta(x-3)) + \frac{2}{15}(\delta(x+\sqrt{6}) + \delta(x-\sqrt{6})).$$

Intersection numbers:

$$c_0 = 3, \quad c_1 = 2, \quad c_2 = 2, \quad c_3 = 2, \quad c_4 = 2, \quad c_5 = 1, \quad c_6 = 1, \quad c_7 = 1,$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 1, \quad b_4 = 1, \quad b_5 = 2, \quad b_6 = 2, \quad b_7 = 2, \quad b_8 = 3.$$

In all of the above examples, a_k is defined in terms of b_k and c_k as

$$a_k = \frac{c_0 c_1 \dots c_{k-1}}{b_1 b_2 \dots b_k}. \quad (\text{C-i})$$

References

- [1] M. A. Jafarizadeh, S. Salimi, e-print quan-ph/0510174.
- [2] A. Nayak and A. Vishwanath, e-print quant-ph/0010117.
- [3] D. Aharonov, A. Ambainis, J. Kempe, and U. Vazirani, in Proceedings of the 33rd ACM Annual Symposium on Theory Computing (ACM Press, New York, 2001).
- [4] Norio Konno, e-print quant-ph/0602213.
- [5] E. Farhi and S. Gutmann, Phys. Rev. A **58**, 915 (1998).
- [6] Y. Aharonov, L. Davidovich, and N. Zagury, Phys. Rev. A **48**, 1687 (1993).
- [7] D. Meyer, J. Stat. Phys. **85**, 551 (1996).
- [8] E. Farhi, M. Childs, and S. Gutmann, Quantum Information Process. **1**, 35 (2002).
- [9] T. A. Brun, H. A. Carteret, and A. Ambainis, Phys. Rev. A **67**, 032304 (2003).
- [10] T. A. Brun, H. A. Carteret, and A. Ambainis, Phys. Rev. A **67**, 052317 (2003).
- [11] T. Yamasaki, H. Kobayashi, and H. Imai, Phys. Rev. A **68**, 012302 (2003).
- [12] J. Kempe, Proceedings of 7th International Workshop on Randomization and Approximation Techniques in Computer Science (RANDOM'03), p. 354 (2003).
- [13] A. Childs, E. Deotto, R. Cleve, E. Farhi, S. Gutmann, D. Spielman, in Proc. 35th Ann. Symp. Theory of Computing (ACM Press), 59(2003).

- [14] N. Shenvi, J. Kempe, and K.B. Whaley, Phys. Rev. A, **67**, 052307 (2003) .
- [15] R. C. Bose and T. Shimamoto (1952), J. Amer. Statist. Assoc. **47**, 151.
- [16] P. Delsarte,(1973), Philips Res. Rep. Suppl. **10**, vi+97.
- [17] P. Terwilliger, J. Algebraic Combin. **1**, 4, 363(1992).
- [18] J.M.P. Balmaceda, M. Oura, J. Math. **48**, 221 (1994).
- [19] E. Bannai, A. Munemasa, J. Math. **49**, 93 (1995).
- [20] K. Tanabe, J. Algebraic Combin. **6**, 173 (1997).
- [21] H. Gerhardt, and J. Watrous, Proceedings of 7th International Workshop on Randomization and Approximation Techniques in Computer Science (RANDOM03), p. 290(2003).
- [22] N. Obata, Interdisciplinary Information Sciences, **10**, 41 (2004).
- [23] R. A. Bailey, *Association Schemes: Designed Experiments, Algebra and Combinatorics* (Cambridge University Press, Cambridge, 2004).
- [24] M. Marcus, and H. Minc, *Introduction to Linear Algebra* (Macmillan, New York 1965).
- [25] John S. Caughman IV, Discrete Mathematics, **196**,65 (1999).
- [26] H. Tarnanen, Europ. J. Combinatorics, **20**, 101 (1999).
- [27] E. Bannai, and T. Ito, *Algebraic Combinatorics I: Association Schemes* (Benjamin Cummings, London, 1984).
- [28] G. James, and M. Liebeck, *Representations and Characters of Groups*(Cambridge University Press 1993).
- [29] M. Tomiyama, Journal of Combinatorial Theory, Series A, **88**, 306(1999).

- [30] C. Moore and A. Russell, in Proceedings of the 6th Int. Workshop on Randomization and Approximation in Computer Science (RANDOM'02)(2002).
- [31] R. Ingram, In Proc. Amer. Math. Soc., **1**, 358 (1950).
- [32] H. Gerhardt, and J. Watrous, Proceedings of 7th International Workshop on Randomization and Approximation Techniques in Computer Science (RANDOM'03), 290 (2003).
- [33] A. Ahmadi, R. Belk, C. Tamon and C. Wendler, Quantum Information and Computation, **3**, 611 (2003).
- [34] E. Egge, J. Algebra, **233**, 213 (2000).
- [35] B. Curtin, Graphs Combin. **15**, 143 (1999).
- [36] A. Hora, and N. Obata, Fundamental Problems in Quantum Physics, World Scientific, **284**(2003).
- [37] Sung Y. Song, Graphs and combinatorics, **18**, 655 (2002).
- [38] J. A. Shohat, and J. D. Tamarkin, *The Problem of Moments*, American Mathematical Society, Providence, RI (1943).
- [39] A. Hora, and N. Obata, Quantum Information V, World Scientific, Singapore (2002).
- [40] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, Science Publishers Inc(1978).

Figure Captions

Figure-1: Shows edges through α and β in a distance regular graph.

Figure-2: Shows the graph of symmetric group S_3 .

Figure-3: The Petersen graph.

Figure-4: The Johnson graph $J(4, 2)$.